

# COLORED-DESCENT REPRESENTATIONS OF COMPLEX REFLECTION GROUPS $G(r, p, n)$

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## ABSTRACT

We study the complex reflection groups  $G(r, p, n)$ . By considering these groups as subgroups of the wreath products  $\mathbb{Z}_r \wr S_n$ , and by using Clifford theory, we define combinatorial parameters and descent representations of  $G(r, p, n)$ , previously known for classical Weyl groups. One of these parameters is the flag major index, which also has an important role in the decomposition of these representations into irreducibles. A Carlitz type identity relating the combinatorial parameters with the degrees of the group, is presented.

## 1. Introduction

Let  $V$  be a complex vector space of dimension  $n$ . A **pseudo-reflection** on  $V$  is a linear transformation on  $V$  of finite order which fixes a hyperplane in  $V$  pointwise. A **complex reflection group** on  $V$  is a finite subgroup  $W$  of  $\mathrm{GL}(V)$

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generated by pseudo-reflections. Such groups are characterized by the structure of their invariant ring. More precisely, let  $\mathbb{C}[V]$  be the symmetric algebra of  $V$  and let us denote by  $\mathbb{C}[V]^W$  the algebra of invariants of  $W$ . Shephard–Todd [25] and Chevalley [13] proved that  $W$  is generated by pseudo-reflections if and only if  $\mathbb{C}[V]^W$  is a polynomial ring.

Irreducible finite complex reflection groups have been classified by Shephard–Todd [25]. In particular, there is a single infinite family of groups and exactly 34 other “exceptional” complex reflection groups. The infinite family  $G(r, p, n)$ , where  $r, p, n$  are positive integers with  $p|r$ , consists of the groups of  $n \times n$  matrices such that:

- 1) the entries are either 0 or  $r^{\text{th}}$  roots of unity;
- 2) there is exactly one nonzero entry in each row and each column;
- 3) the  $(r/p)^{\text{th}}$  power of the product of the nonzero entries is 1.

The classical Weyl groups appear as special cases;  $G(1, 1, n) = S_n$  the symmetric group,  $G(2, 1, n) = B_n$  the hyperoctahedral group and  $G(2, 2, n) = D_n$  the group of even-signed permutations.

Through research on complex reflection groups and their braid groups and Hecke algebras, the fact that complex reflection groups behave like Weyl groups has become more and more clear. In particular, it has been recently discovered that they (and not only Weyl groups) play a key role in the structure as well as in the representation theory of finite reductive groups. For more information on these results the reader is advised to consult the survey article of Broué [9], and the handbook of Geck and Malle [11].

One of the aims of this paper is to show that complex reflection groups continue to behave like (classical) Weyl groups even from the point of view of the combinatorial representation theory. It is well-known that, in a way similar to Coxeter groups, they have presentations in terms of generators and relations, that can be visualized by Dynkin type diagrams (see e.g., [10]). Moreover, their elements can be represented as colored permutations. In fact, the complex reflection groups  $G(r, p, n)$  can naturally be identified as normal subgroups of index  $p$  of the wreath product  $G(r, n) := \mathbb{Z}_r \wr S_n$ , where  $\mathbb{Z}_r$  is the cyclic group of order  $r$ . This makes it possible to handle them by purely combinatorial methods. We follow this approach.

The organization of the paper is as follows. In Section 3, we introduce several combinatorial parameters on complex reflection groups. Among those we define the concepts of “major index” and “descent number” for  $G(r, p, n)$ . Then we continue our investigation by showing the interplay between these new combi-

natorial statistics and the representation theory of  $G(r, p, n)$ . More precisely, in Section 8 we define a new set of  $G(r, p, n)$ -modules, which we call **colored-descent representations**. They generalize to all groups  $G(r, p, n)$ , the descent representations introduced for  $S_n$  and  $B_n$ , by Adin, Brenti, and Roichman [3]. These modules are isomorphic to the Solomon’s descent representations in the case of  $S_n$ , while they decompose Solomon’s representations in the case of  $B_n$ . Following Adin–Brenti–Roichman’s approach, we use the coinvariant algebra as representation space. In order to do that, in Section 4, we use refined statistics to define an explicit monomial basis for the coinvariant space. This basis has special properties that allow us to define our new set of modules. In Section 10, the decomposition into irreducibles of the colored-descent representations is provided (Theorem 10.5). We use a generalization of a formula of Stanley (Theorem 9.3) on a specialization of a product of Schur functions, proved in Section 9. It turns out that the multiplicity of any irreducible representations is counted by the cardinality of a particular class of standard Young tableaux, called  **$n$ -orbital**. As a corollary of that, we obtain a refinement of a theorem first attributed to Stembridge (Corollary 10.6). Finally, in Section 11, by using the properties of the colored descent basis, a Carlitz type identity (Theorem 11.2) relating the major index and descent number with the degrees of  $G(r, p, n)$  will be derived.

It is worth mentioning, that in a very recent preprint concerning wreath products, Baumann and Hohlweg [6], following Solomon’s descent algebra approach [27], define some special characters of  $G(r, n)$  as images, through a “generalized Solomon homomorphism”, of the elements of a basis of the Mantaci–Reutenaur algebra [21]. Nevertheless, they don’t provide any module having them as characters. It was a nice surprise to see that the modules which carry them as characters are our colored-descent representations in the case of  $G(r, 1, n)$ .

## 2. Complex Reflection Groups $G(r, p, n)$

For our exposition it will be much more convenient to consider wreath products not as groups of complex matrices, but as groups of colored permutations.

Let  $\mathbb{P} := \{1, 2, \dots\}$ ,  $\mathbb{N} := \mathbb{P} \cup \{0\}$ , and  $\mathbb{C}$  be the field of complex numbers. For any  $n \in \mathbb{P}$ , we let  $[n] := \{1, 2, \dots, n\}$ . Let  $S_n$  be the symmetric group on  $[n]$ . A permutation  $\sigma \in S_n$  will be denoted by  $\sigma = \sigma(1) \cdots \sigma(n)$ .

Let  $r, n \in \mathbb{P}$ . The **wreath product**  $G(r, n)$  of  $\mathbb{Z}_r$  by  $S_n$  is defined by

$$(1) \quad G(r, n) := \{((c_1, \dots, c_n), \sigma) \mid c_i \in [0, r-1], \sigma \in S_n\}.$$

Any  $c_i$  can be considered as the color of the corresponding entry  $\sigma(i)$ . This explains the fact that this group is also called the **group of r-colored permutations**. Sometimes we will represent its elements in **window notation** as

$$g = g(1) \cdots g(n) = \sigma(1)^{c_1} \cdots \sigma(n)^{c_n}.$$

When it is not clear from the context, we will denote  $c_i$  by  $c_i(g)$ . Moreover, if  $c_i = 0$ , it will be omitted in the window notation of  $g$ . We denote by

$$\text{Col}(g) := (c_1, \dots, c_n) \quad \text{and} \quad \text{col}(g) := \sum_{i=1}^n c_i,$$

the **color vector** and the **color weight** of any  $g := ((c_1, \dots, c_n), \sigma) \in G(r, n)$ . For example, for  $g = 4^1 3 2^4 1^2 \in G(5, 4)$  we have  $\text{Col}(g) = (1, 0, 4, 2)$  and  $\text{col}(g) = 7$ .

Now let  $p \in \mathbb{P}$  be such that  $p|r$ . The **complex reflection group**  $G(r, p, n)$  is the subgroup of  $G(r, n)$  defined by

$$(2) \quad G(r, p, n) := \{g \in G(r, n) \mid \text{col}(g) \equiv 0 \pmod{p}\}.$$

Note that  $G(r, p, n)$  is the kernel of the map  $G(r, n) \longrightarrow \mathbb{Z}_p$ , sending  $g$  to its color weight  $\text{col}(g)$ , and so it is a normal subgroup of  $G(r, n)$  of index  $p$ . It is clear from the definition that the wreath product is  $G(r, 1, n)$ . Moreover,  $G(1, 1, n)$  is the symmetric group,  $G(2, 2, n)$  is the Weyl groups of type  $D$ , and  $G(r, r, 2)$  is the dihedral group of order  $2r$ .

### 3. Flag Major Index on $G(r, p, n)$

The major index is a very well studied statistic over the symmetric group, (see e.g., [12], [14], [15], [17], [20]). Lately, many generalizations of this concept have been given for the other classical Weyl groups and for wreath products (see, e.g., [1], [2], [7], [23], [30]). Among those statistics, it became soon clear that, the flag major index, introduced by Adin and Roichman on  $B_n$  [1], is the one more similar to the major index [15], regarding the aspect of its implications with the representation theory of the group [1, 3]. Its analogue for Weyl groups of type  $D$  is the  $D$ -(flag) major index introduced in [7]. In this section we define a version of the major index for  $G(r, p, n)$ . We introduce a particular subset of  $G(r, n)$  which is in bijective correspondence with  $G(r, p, n)$ , and define our statistics there. In the following sections, it will become clear that this subset is the right object to work with.

In order to lighten the notation, we let  $G := G(r, n)$  and  $H := G(r, p, n)$ . For any  $r, p, n \in \mathbb{P}$ , with  $p|r$  let  $d := r/p$ . We define the following subset of  $G(r, n)$ ,

$$(3) \quad \Gamma(r, p, n) = \{\gamma = ((c_1, \dots, c_n), \sigma) \in G(r, n) \mid c_n < d\}.$$

Note that  $\Gamma := \Gamma(r, p, n)$  is not a subgroup of  $G$ . Clearly,  $|\Gamma| = n!r^{n-1}d$  and so it is in bijection with  $H$ . Moreover, one can easily check that the mapping  $\varphi$

$$(4) \quad ((c_1, \dots, c_n), \sigma) \xrightarrow{\varphi} \left( (c_1, \dots, \lfloor \frac{c_n}{p} \rfloor), \sigma \right)$$

is a bijection between  $H$  and  $\Gamma$ . As usual, for any  $a \in \mathbb{Q}$ ,  $\lfloor a \rfloor$  denotes the greatest integer  $\leq a$ .

In order to make our definitions more natural and clear, from now on, we will work with  $\Gamma$  instead of  $H$ . Clearly, via the above bijection  $\varphi$ , every function on  $\Gamma$  can be considered as a function on  $H$  and vice versa.

We fix the following order  $\prec$  on colored integer numbers

$$(5) \quad 1^{r-1} \prec 2^{r-1} \prec \dots \prec n^{r-1} \prec \dots \prec 1^1 \prec 2^1 \prec \dots \prec n^1 \prec 1 \prec 2 \prec \dots \prec n.$$

The **descent set** of a colored integer sequence  $\gamma \in \Gamma$  is defined by

$$\text{Des}(\gamma) := \{i \in [n-1] : \gamma_i \succ \gamma_{i+1}\}.$$

The **major index** of  $\gamma$  is the sum of all the descents of  $\gamma$ , i.e.,

$$\text{maj}(\gamma) := \sum_{i \in \text{Des}(\gamma)} i.$$

Following [1], we define the **flag major index** of  $\gamma \in \Gamma$  by

$$\text{fmaj}(\gamma) := r \cdot \text{maj}(\gamma) + \text{col}(\gamma).$$

Via the identification given by the bijection  $\varphi$  we consider this as the flag major index on complex reflection groups. Often, for our purposes, it will be better to obtain this statistic in a more elaborate way. For any  $\gamma = ((c_1, \dots, c_n), \sigma) \in \Gamma$  let

$$(6) \quad d_i(\gamma) := |\{j \in \text{Des}(\gamma) : j \succeq i\}| \text{ and}$$

$$(7) \quad f_i(\gamma) := r \cdot d_i(\gamma) + c_i(\gamma).$$

For any  $\gamma \in \Gamma$ , we define the **flag descent number** of  $\gamma$  by

$$(8) \quad \text{fdes}(\gamma) := r \cdot d_1(\gamma) + c_1.$$

It is clear that for every  $\gamma \in \Gamma$ ,  $\text{fmaj}(\gamma) := \sum_{i=1}^n f_i(\gamma)$ , and that  $d_1(\gamma)$  is the cardinality of the set  $\text{Des}(\gamma)$ . Let us conclude this section with an example.

*Example 3.1:* Let  $\gamma = \mathbf{6}2^54^4\mathbf{3}^11^65^3 \in \Gamma(8, 2, 6)$ . The set of descents is  $\text{Des}(\gamma) = \{1, 4\}$ . Hence  $(d_1(\gamma), \dots, d_n(\gamma)) = (2, 1, 1, 1, 0, 0)$ ,  $(f_1(\gamma), \dots, f_n(\gamma)) = (16, 13, 12, 9, 6, 3)$ , and so  $\text{fmaj}(\gamma) = 59$  and  $\text{fdes}(\gamma) = 8$ .

#### 4. Colored Descent Basis

Let  $W \leq \text{GL}(V)$  be a complex reflection group. If we set  $\mathbf{x} = x_1, \dots, x_n$  as a basis for  $V$ , then  $\mathbb{C}[V]$  can be identified with the ring of polynomials  $\mathbb{C}[\mathbf{x}]$ . The ring of invariants  $\mathbb{C}[\mathbf{x}]^W$  is then generated by 1 and a set of  $n$  algebraically independent homogeneous polynomials  $\{\vartheta_1, \dots, \vartheta_n\}$  which are called **basic invariants**. Although these polynomials are not uniquely determined, their degrees  $d_1, \dots, d_n$  are basic numerical invariants of the group, and are called the **degrees of  $W$** . Let us denote by  $\mathcal{I}_W$  the ideal generated by the invariants of strictly positive degree. The **coinvariant space** of  $W$  is defined by

$$\mathbb{C}[\mathbf{x}]_W := \mathbb{C}[\mathbf{x}] / \mathcal{I}_W.$$

Since  $\mathcal{I}_W$  is  $W$ -invariant, the group  $W$  acts naturally on  $\mathbb{C}[\mathbf{x}]_W$ . It is well-known that  $\mathbb{C}[\mathbf{x}]_W$  is isomorphic to the **left regular representation** of  $W$ . It follows that its dimension as a  $\mathbb{C}$ -module is equal to the order of the group  $W$ .

As a first application of the flag major index, we find a basis of the coinvariant space of  $H$ . The wreath product  $G$  acts on the ring of polynomials  $\mathbb{C}[\mathbf{x}]$  as follows;

$$(9) \quad \sigma(1)^{c_1} \cdots \sigma(n)^{c_n} \cdot P(x_1, \dots, x_n) = P(\zeta^{c_{\sigma(1)}} x_{\sigma(1)}, \dots, \zeta^{c_{\sigma(n)}} x_{\sigma(n)}),$$

where  $\zeta$  denotes a primitive  $r^{\text{th}}$  root of the unity. A set of basic invariants under this action is given by the **elementary symmetric functions**  $e_j(x_1^r, \dots, x_n^r)$ ,  $1 \leq j \leq n$ . Now, consider the restriction of the previous action on  $\mathbb{C}[\mathbf{x}]$  to  $H$ . A set of fundamental invariants is given by

$$\vartheta_j(x_1, \dots, x_n) := \begin{cases} e_j(x_1^r, \dots, x_n^r) & \text{for } j = 1, \dots, n-1 \\ x_1^d \cdots x_n^d & \text{for } j = n. \end{cases}$$

It follows that the degrees of  $H$  are  $r, 2r, \dots, (n-1)r, nd$ . Let  $\mathcal{I}_H := (\vartheta_1, \dots, \vartheta_n)$  be the ideal generated by the constant term invariant polynomials of  $H$ . The dimension of the coinvariant space,  $\mathbb{C}[\mathbf{x}]_H := \mathbb{C}[\mathbf{x}] / \mathcal{I}_H$ , is equal to  $|H|$  that is,  $n!r^n/p$ . In what follows we will associate to any element  $\gamma \in \Gamma$  an ad-hoc monomial in  $\mathbb{C}[\mathbf{x}]$ . Those monomials will form a linear basis of  $\mathbb{C}[\mathbf{x}]_H$ .

Let  $\gamma = ((c_1, \dots, c_n), \sigma) \in \Gamma$ . We define

$$(10) \quad \mathbf{x}_\gamma := \prod_{i=1}^n x_{\sigma(i)}^{f_i(\gamma)}.$$

By definition  $f_n(\gamma) < d$ , hence  $\mathbf{x}_\gamma$  is nonzero in  $\mathbb{C}[\mathbf{x}]_H$ . Clearly  $\deg(\mathbf{x}_\gamma) = \text{fmaj}(\gamma)$ .

For example, let  $\gamma = 62^5 4^4 3^1 1^6 5^3 \in G(8, 2, 6)$  as in Example 3.1. The associated monomial is  $\mathbf{x}_\gamma = x_1^6 x_2^{13} x_3^9 x_4^{12} x_5^3 x_6^{16}$ .

We restrict our attention to the quotient  $S := \mathbb{C}[\mathbf{x}]/(\vartheta_n)$ . Hence we consider nonzero monomials  $M = \prod_{i=1}^n x_i^{a_i}$  such that  $a_i < d$  for at least one  $i \in [n]$ . We associate to  $M$  the unique element  $\gamma(M) = ((c_1, \dots, c_n), \sigma) \in \Gamma$  such that for all  $i \in [n]$ :

- i)  $a_{\sigma(i)} \geq a_{\sigma(i+1)}$ ;
- ii)  $a_{\sigma(i)} = a_{\sigma(i+1)} \implies \sigma(i) < \sigma(i+1)$ ;
- iii)  $a_{\sigma(i)} \equiv c_i \pmod{r}$ .

We denote by  $\lambda(M) := (a_{\sigma(1)}, \dots, a_{\sigma(n)})$  the **exponent partition** of  $M$ , and we call  $\gamma(M) \in \Gamma$  the **colored index permutation**.

Now, let  $M = \prod_{i=1}^n x_i^{a_i}$  be a nonzero monomial in  $S$ , and let  $\gamma := \gamma(M)$  be its colored index permutation. Consider the monomial  $\mathbf{x}_\gamma$  associated to  $\gamma$ . It is not hard to see that the sequence  $(a_{\sigma(i)} - f_i(\gamma)), i = 1, \dots, n-1$ , of exponents of  $M/\mathbf{x}_\gamma$ , consists of nonnegative integers of the form  $c \cdot r$  with  $c > 0$ , and is weakly decreasing. This allows us to associate to  $M$  the **complementary partition**  $\mu(M)$ , defined by

$$(11) \quad \mu'(M) := \left( \frac{a_{\sigma(i)} - f_i(\gamma)}{r} \right)_{i=1}^{n-1},$$

where, as usual,  $\mu'$  denotes the conjugate partition of  $\mu$ .

*Example 4.1:* Let  $r = 8$ ,  $p = 2$ , and  $n = 6$  and consider the monomial

$$M = x_1^6 x_2^{21} x_3^{17} x_4^{20} x_5^3 x_6^{32} \in \mathbb{C}[x_1, \dots, x_6]/(\vartheta_6).$$

The exponent partition  $\lambda(M) = (32, 21, 20, 17, 6, 3)$  is obtained by reordering the power of  $x_i$ 's following the colored index permutation  $\gamma(M) = 62^5 4^4 3^1 1^6 5^3 \in \Gamma(8, 2, 6)$ . We have already computed the monomial  $\mathbf{x}_{\gamma(M)} = x_1^6 x_2^{13} x_3^9 x_4^{12} x_5^3 x_6^{16}$ . It follows that  $\mu(M) = (4, 1)$ .

Now we define a partial order  $\sqsubseteq$  on the monomials of the same total degree in  $S$ . Let  $M$  and  $M'$  be nonzero monomials in  $S$  with the same total degree and such that the exponents of  $x_i$  in  $M$  and  $M'$  have the same parity (mod  $r$ ) for every  $i \in [n]$ . Then we write  $M' \sqsubset M$  if one of the following holds:

- 1)  $\lambda(M') \triangleleft \lambda(M)$ , or
- 2)  $\lambda(M') = \lambda(M)$  and  $\text{inv}(\gamma(M')) > \text{inv}(\gamma(M))$ .

Here,  $\text{inv}(\gamma) := |\{(i, j) | i < j \text{ and } \gamma(i) \succ \gamma(j)\}|$ , and  $\triangleleft$  denotes the **dominance order** defined on the set of partitions of a fixed nonnegative integer  $n$  by:  $\mu \trianglelefteq \lambda$  if for all  $i \geq 1$

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

By arguments similar to those given in the proof of Lemma 3.3 in [8], it can be showed that every monomial  $M \in S$  admits the following expansion in terms of the basis monomials  $\mathbf{x}_\gamma$ 's and the basic generators  $\{\vartheta_i\}$  of the ideal  $\mathcal{I}_H$ ;

$$(12) \quad M = \vartheta_{\mu(M)} \cdot \mathbf{x}_{\gamma(M)} + \sum_{M' \sqsubset M} n_{M', M} \vartheta_{\mu(M')} \cdot \mathbf{x}_{\gamma(M')},$$

where  $n_{M, M'}$  are integers. As usual,  $\vartheta_\mu := \vartheta_{\mu_1} \vartheta_{\mu_2} \cdots \vartheta_{\mu_\ell}$ , with  $\ell := \ell(\mu)$  the **length** of the partition  $\mu$ .

*Example 4.2:* Consider  $\Gamma(6, 2, 3)$  and the monomial  $M = x_1^{11} x_2^8 x_3$ . We have  $\gamma(M) = 1^5 2^2 3^1 \in \Gamma(6, 2, 3)$ ,  $\mathbf{x}_{\gamma(M)} = x_1^5 x_2^2 x_3$ , and  $\mu(M) = (2)$ . Then, if we set  $M_1 = x_1^{11} x_2^7 x_3$ , and  $M_2 = x_1^5 x_2^8 x_3^7$  we have that  $M = \mathbf{x}_{\gamma(M)} \vartheta_2 - M_1 - M_2$ . It is easy to see that  $M_1, M_2 \sqsubset M$  and that  $M_1 = \mathbf{x}_{\gamma(M_1)}$ , and  $M_2 = \mathbf{x}_{\gamma(M_2)}$ , then

$$M = \mathbf{x}_{\gamma(M)} \vartheta_2 - \mathbf{x}_{\gamma(M_1)} - \mathbf{x}_{\gamma(M_2)}.$$

Since the monomials  $\{\mathbf{x}_\gamma + \mathcal{I}_H \mid \gamma \in \Gamma\}$  are generators for  $\mathbb{C}[\mathbf{x}]_H$ , and  $\dim \mathbb{C}[\mathbf{x}]_H = |H|$ , they form a basis, called the **colored-descent basis**. We summarize this by

**THEOREM 4.3:** *The set*

$$\{\mathbf{x}_\gamma + \mathcal{I}_H \mid \gamma \in \Gamma\}$$

*is a basis for  $\mathbb{C}[\mathbf{x}]_H$ .*

Note that, when  $H$  is specialized to one of the classical Weyl groups, our basis coincides with the descent basis defined by Garsia–Stanton for  $S_n$ , [16], by Adin–Brenti–Roichman for  $B_n$ , [3] and by Biagioli–Caselli for  $D_n$ , [8]. Recently, another basis for  $\mathbb{C}[\mathbf{x}]_H$  has been given by Allen, [4]. Although both our and Allen's basis coincide with the Garsia–Stanton basis in the case of  $S_n$ , in general they are different, as can be checked even for the small case of  $G(2, 2, 2)$ . It would be interesting to see if Allen's basis leads to an analogous definition of descent representations (see Section 8).



*Example 4.4:* The elements of  $\Gamma(6, 3, 2)$ ,  $d = 2$ , are

$$\begin{array}{cccccc} 12 & 1^1 2 & 1^2 2 & 1^3 2 & 1^4 2 & 1^5 2 \\ 12^1 & 1^1 2^1 & 1^2 2^1 & 1^3 2^1 & 1^4 2^1 & 1^5 2^1 \\ 21 & 2^1 1 & 2^2 1 & 2^3 1 & 2^4 1 & 2^5 1 \\ 21^1 & 2^1 1^1 & 2^2 1^1 & 2^3 1^1 & 2^4 1^1 & 2^5 1^1. \end{array}$$

The corresponding monomials

$$\begin{array}{cccccc} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ x_1^6 x_2 & x_1 x_2 & x_1^2 x_2 & x_1^3 x_2 & x_1^4 x_2 & x_1^5 x_2 \\ x_2^6 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ x_2^6 x_1 & x_2^7 x_1 & x_2^2 x_1 & x_2^3 x_1 & x_2^4 x_1 & x_2^5 x_1 \end{array}$$

form a basis for  $\mathbb{C}[x_1, x_2]/(x_1^6 + x_2^6, x_1^2 x_2^2)$ .

## 5. The Representation Theory of $G(r, p, n)$

In this section we present the representation theory of the group  $H := G(r, p, n)$ . We follow the exposition of [31] (see also [22]). Since the irreducible representations of  $H$  are related to the irreducible representations of  $G$  via Clifford Theory, we start this section by presenting the representation theory of  $G$ .

Let  $g = \sigma(1)^{c_1} \cdots \sigma(n)^{c_n} \in G$ . First divide  $\sigma \in S_n$  into cycles, and then provide the entries with their original colors  $c_i$ , thus obtaining **colored cycles**. The color of a cycle is simply the sum of all the colors of its entries. For every  $i \in [0, r-1]$ , let  $\alpha^i$  be the partition formed by the lengths of the cycles of  $g$  having color  $i$ . We may thus associate  $g$  with the  $r$ -partition  $\vec{\alpha} = (\alpha^0, \dots, \alpha^{r-1})$ . Note that  $\sum_{i=0}^{r-1} |\alpha^i| = n$ . We refer to  $\vec{\alpha}$  as the **type** of  $g$ . For example, the decompositions into colored cycles of  $\gamma = 62^5 4^4 3^1 1^6 5^3 \in G(8, 6)$  is  $(1^6 65^3)(3^1 4^4)(2^5)$ . We have  $\alpha^1 = (3)$ ,  $\alpha^5 = (2, 1)$ , and  $\alpha^i = 0$  for all other  $i$ 's.

One can prove that two elements of  $G$  are conjugate if and only if they have the same type. It is well-known that irreducible representations of  $G$  are also indexed by  $r$ -tuple of partitions  $\vec{\lambda} := (\lambda^0, \dots, \lambda^{r-1})$  with  $\sum_{i=0}^{r-1} |\lambda^i| = n$ . We denote this set by  $\mathcal{P}_{r,n}$ .

As mentioned above, the passage to the representation theory of  $H$ , is by Clifford theory. The group  $G/H$  can be identified with the cyclic group  $C$  of order  $p$  consisting of the characters  $\delta$  of  $G$  satisfying  $H \subset \text{Ker}(\delta)$ . More precisely, define the linear character  $\delta_0$  of  $G$  by  $\delta_0((c_1, \dots, c_n), \sigma) := \zeta^{c_1 + \cdots + c_n}$ , so that  $C = \langle \delta_0^d \rangle \simeq \mathbb{Z}_p$ .

The group  $C$  acts on the set of irreducible representations of  $G$  by  $V(\vec{\lambda}) \mapsto \delta \otimes V(\vec{\lambda})$ , where  $V(\vec{\lambda})$  is the irreducible representation of  $G$  indexed

by  $\vec{\lambda}$ , and  $\delta \in C$ . This action can be explicitly described as follows. Let  $\vec{\lambda} = (\lambda^0, \dots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$ . Define a **1-shift** of  $\vec{\lambda}$  by

$$(13) \quad (\vec{\lambda})^{\circ 1} := (\lambda^{r-1}, \lambda^0, \dots, \lambda^{r-2}).$$

By applying  $i$ -times the **shift operator** we get  $(\vec{\lambda})^{\circ i}$ . Then one can show (see [22, Section 4]) that

$$(14) \quad \delta_0 \otimes V(\vec{\lambda}) \simeq V((\vec{\lambda})^{\circ 1}),$$

for every  $\vec{\lambda} \in \mathcal{P}_{r,n}$ . Now let us denote by  $[\vec{\lambda}]$  a  $C$ -orbit of the representation  $V(\vec{\lambda})$ . From (14) we obtain that  $[\vec{\lambda}] = \{V(\vec{\mu}) : \vec{\mu} \sim \vec{\lambda}\}$ , where the equivalence relation is defined by

$$(15) \quad \vec{\lambda} \sim \vec{\mu} \text{ if and only if } \vec{\mu} = (\vec{\lambda})^{\circ i \cdot d} \text{ for some } i \in [0, p-1].$$

Let us denote  $b(\vec{\lambda}) := |[ \vec{\lambda} ]|$ , and set  $u(\vec{\lambda}) := p/b(\vec{\lambda})$ . Consider the stabilizer of  $\vec{\lambda}$ ,  $C_{\vec{\lambda}}$ , that is

$$C_{\vec{\lambda}} := \{\delta \in C \mid V(\vec{\lambda}) = \delta \otimes V(\vec{\lambda})\}.$$

Clearly,  $C_{\vec{\lambda}}$  is a subgroup of  $C$  generated by  $\delta_0^{b(\vec{\lambda}) \cdot d}$  and so  $|C_{\vec{\lambda}}| = u(\vec{\lambda})$ .

It can be proven that the restriction of the irreducible representation  $V(\vec{\lambda})$  of  $G$  to  $H$ , decomposes into  $u(\vec{\lambda}) = |C_{\vec{\lambda}}|$  non-isomorphic irreducible  $H$  modules. On the other hand, any other  $G$ -module in the same orbit  $[\vec{\lambda}]$  will give us the same result. Actually, one can prove even more (see e.g., [31]).

**THEOREM 5.1:** *There is a one to one correspondence between the irreducible representations of  $H$  and the ordered pairs  $([\vec{\lambda}], \delta)$  where  $[\vec{\lambda}]$  is the orbit of the irreducible representation  $V(\vec{\lambda})$  of  $G$  and  $\delta \in C_{\vec{\lambda}}$ . Moreover, if  $\chi^{\vec{\lambda}} \downarrow_H^G$  denotes the restriction of the character of  $V(\vec{\lambda})$  to  $H$ , then*

- i)  $\chi^{\vec{\lambda}} \downarrow_H^G = \chi^{\vec{\mu}} \downarrow_H^G$ , for all  $\vec{\lambda} \sim \vec{\mu}$  and
- ii)  $\chi^{\vec{\lambda}} \downarrow_H^G = \sum_{\delta \in C_{\vec{\lambda}}} \chi^{([\vec{\lambda}], \delta)}$ .

Here is a simple but important example. The irreducible representations of  $B_n$  ( $G(2, 1, n)$  in our notation) are indexed by bi-partitions of  $n$ . The Coxeter group  $D_n$  ( $G(2, 2, n)$  in our notation), is a subgroup of  $B_n$  of index 2. Thus the stabilizer of the action of  $B_n/D_n \cong \mathbb{Z}_2$  on a pair of Young diagrams  $\vec{\lambda} = (\lambda^1, \lambda^2)$  is either  $\mathbb{Z}_2$  if  $\lambda^1 = \lambda^2$ , or  $\{\text{id}\}$  if  $\lambda^1 \neq \lambda^2$ . In the first case, the irreducible representation of  $B_n$  corresponding to  $\vec{\lambda}$ , when restricted to  $D_n$ , splits into two nonisomorphic irreducible representations of  $D_n$ . In the second case  $\vec{\lambda} = (\lambda^1, \lambda^2)$  and  $\vec{\lambda}^T = (\lambda^2, \lambda^1)$  correspond to two isomorphic irreducible representations of  $D_n$ .

## 6. Frobenius Formula for $G(r, p, n)$

Let  $\mathbf{y} = y_1, y_2, \dots$ , be an infinite set of variables and let  $\lambda$  be a partition of  $n$ . Let us denote by  $p_\lambda(\mathbf{y}) := p_{\lambda_1}(\mathbf{y})p_{\lambda_2}(\mathbf{y}) \cdots p_{\lambda_\ell}(\mathbf{y})$  the **power sum symmetric function**, where

$$p_k(\mathbf{y}) := y_1^k + y_2^k + \cdots.$$

A well-known theorem of Frobenius establishes a link between the power sum and the Schur functions via the characters of the symmetric group. Let us denote by  $\chi_\alpha^\lambda$  the character of the irreducible representation of  $S_n$  corresponding to  $\lambda$  calculated in the conjugacy class  $\alpha$ , and by  $s_\lambda(\mathbf{y})$  the **Schur function** corresponding to  $\lambda$ . Then we have (see e.g., [28, Corollary 7.17.4])

$$p_{\vec{\alpha}}(\mathbf{y}) = \sum_{\lambda \vdash n} \chi_{\vec{\alpha}}^\lambda s_\lambda(\mathbf{y}).$$

We need an analogous formula involving the characters of the group  $H$ . In developing such an analogue, we follow [19, Appendix B] but we use a slightly different notation.

Let  $\vec{\alpha} = (\alpha^0, \dots, \alpha^{r-1}) \in \mathcal{P}_{r,n}$  be an  $r$ -partition of  $n$  and let  $\mathbf{y}^0, \dots, \mathbf{y}^{r-1}$  be  $r$  sets of independent variables. We define

$$(16) \quad p_{\vec{\alpha}}(\mathbf{y}^0, \dots, \mathbf{y}^{r-1}) := \prod_{j=0}^{r-1} \prod_{i=1}^{\ell(\alpha^j)} \sum_{k=0}^{r-1} \zeta^{j \cdot k} p_{\alpha_i^j}(\mathbf{y}^k),$$

where  $\zeta$  is a primitive  $r^{\text{th}}$  root of the unity. For example, for  $r = 4$  we get:

$$\begin{aligned} & p_{(\alpha^0, \alpha^1, \alpha^2, \alpha^3)}(\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3) \\ &= \prod_{i=1}^{\ell(\alpha^0)} (p_{\alpha_i^0}(\mathbf{y}^0) + p_{\alpha_i^0}(\mathbf{y}^1) + p_{\alpha_i^0}(\mathbf{y}^2) + p_{\alpha_i^0}(\mathbf{y}^3)) \\ &\quad \times \prod_{i=1}^{\ell(\alpha^1)} (p_{\alpha_i^1}(\mathbf{y}^0) + \zeta p_{\alpha_i^1}(\mathbf{y}^1) + \zeta^2 p_{\alpha_i^1}(\mathbf{y}^2) + \zeta^3 p_{\alpha_i^1}(\mathbf{y}^3)) \\ &\quad \times \prod_{i=1}^{\ell(\alpha^2)} (p_{\alpha_i^2}(\mathbf{y}^0) + \zeta^2 p_{\alpha_i^2}(\mathbf{y}^1) + \zeta^0 p_{\alpha_i^2}(\mathbf{y}^2) + \zeta^2 p_{\alpha_i^2}(\mathbf{y}^3)) \\ &\quad \times \prod_{i=1}^{\ell(\alpha^3)} (p_{\alpha_i^3}(\mathbf{y}^0) + \zeta^3 p_{\alpha_i^3}(\mathbf{y}^1) + \zeta^2 p_{\alpha_i^3}(\mathbf{y}^2) + \zeta p_{\alpha_i^3}(\mathbf{y}^3)). \end{aligned}$$

Now, for any  $\vec{\lambda}$  and  $\vec{\alpha}$  in  $\mathcal{P}_{r,n}$ , let  $\chi_{\vec{\alpha}}^{\vec{\lambda}}$  be the value of the irreducible character of  $G$  indexed by  $\vec{\lambda}$  on the conjugacy class of type  $\vec{\alpha}$ . The following theorem

presents the analogue of Frobenius formula for  $G$ . Its proof can be deduced by the proof of formula 9.5" in [19, page 177]. See also [26].

**THEOREM 6.1:** *Let  $\vec{\alpha} \in \mathcal{P}_{r,n}$  be an  $r$ -partition of  $n$ . Then*

$$p_{\vec{\alpha}}(\mathbf{y}^0, \dots, \mathbf{y}^{r-1}) = \sum_{\vec{\lambda} \in \mathcal{P}_{r,n}} \chi_{\vec{\alpha}}^{\vec{\lambda}} s_{\lambda^0}(\mathbf{y}^0) \cdots s_{\lambda^{r-1}}(\mathbf{y}^{r-1}).$$

From Theorem 5.1 and Theorem 6.1 we obtain the following identity.

**THEOREM 6.2** (Frobenius Formula for  $H$ ): *Let  $\vec{\alpha} \in \mathcal{P}_{r,n}$ . Then*

$$p_{\vec{\alpha}}(\mathbf{y}^0, \dots, \mathbf{y}^{r-1}) = \sum_{[\vec{\lambda}]} \left( \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_1)} + \cdots + \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_{u(\vec{\lambda})})} \right) \cdot \sum_{\vec{\mu} \in [\vec{\lambda}]} s_{\mu^0}(\mathbf{y}^0) \cdots s_{\mu^{r-1}}(\mathbf{y}^{r-1}),$$

where  $[\vec{\lambda}]$  runs over all the orbits of irreducible  $G$ -modules,  $u(\vec{\lambda}) = |C_{\vec{\lambda}}|$ , and  $\vec{\mu}$  runs over the elements of the orbit  $[\vec{\lambda}]$ .

## 7. $n$ -Orbital Standard Tableaux

In this section we introduce a new class of standard Young  $r$ -tableaux that will be useful for our purposes later on.

Let  $\vec{\lambda} = (\lambda^0, \dots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$  be an  $r$ -partition of  $n$ . A **Ferrers diagram** of shape  $\vec{\lambda}$  is obtained by the union of the Ferrers diagrams of shapes  $\lambda^0, \dots, \lambda^{r-1}$ , where the  $(i+1)^{\text{th}}$  diagram lies south west of the  $i^{\text{th}}$ . A **standard Young  $r$ -tableau**  $T := (T^0, \dots, T^{r-1})$  of shape  $\vec{\lambda}$  is obtained by inserting the integers  $1, 2, \dots, n$  as **entries** in the corresponding Ferrers diagram increasing along rows and down columns of each diagram separately. We denote by  $\text{SYT}(\vec{\lambda})$  the set of all  $r$ -standard Young tableaux of shape  $\vec{\lambda}$ . Any entry in the  $i$ -th component  $T^i$  of  $T \in \text{SYT}(\vec{\lambda})$  will be considered as having color  $i$ .

A **descent** in an  $r$ -standard Young tableau  $T$  is an entry  $i$  such that  $i+1$  is strictly below  $i$ . We denote the set of descents in  $T$  by  $\text{Des}(T)$ . Similarly to Section 3 we let

$$\begin{aligned} d_i(T) &:= |\{j \geq i \mid j \in \text{Des}(T)\}|, c_i = c_i(T) := k \text{ if } i \in T^k; \\ f_i(T) &:= r \cdot d_i(T) + c_i(T), f(T) := (f_1(T), \dots, f_n(T)) \\ \text{col}(T) &:= c_1 + \cdots + c_n, \\ \text{maj}(T) &:= \sum_{i \in \text{Des}(T)} i, \text{ and } \text{fmaj}(T) := r \cdot \text{maj}(T) + \text{col}(T). \end{aligned}$$

The tableau  $T_1$  in Figure 1 belongs to  $\text{SYT}((1), (2), (2, 1), (1, 1), (3, 1), (2))$ , for example. We have that  $\text{Des}(T) = \{1, 3, 5, 8, 11, 12\}$ ,  $\text{maj}(T) = 40$  and  $\text{col}(T) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 4 + 5 \cdot 2 = 40$ , so  $\text{fmaj}(T) = 280$ .

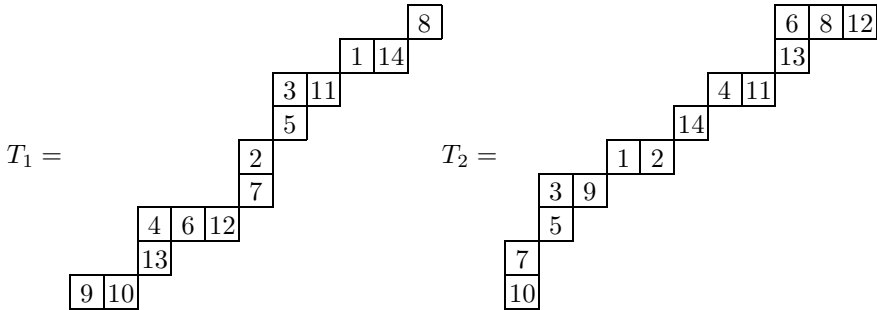


Figure 1. Two 6-standard Young tableaux

Let  $\vec{\lambda} = (\lambda^0, \dots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$ . As in Section 5 let  $[\vec{\lambda}] = \{\vec{\mu} \in \mathcal{P}_{r,n} \mid \vec{\mu} \sim \vec{\lambda}\}$  be the orbit of  $\vec{\lambda}$  under the equivalence relation  $\sim$  defined in (15). An **orbital standard Young tableau**  $T = (T^0, \dots, T^{r-1})$  of **type**  $[\vec{\lambda}]$  is a standard Young  $r$ -tableau having one of the shapes in  $[\vec{\lambda}]$ . The following definition is fundamental in our work. An  **$n$ -orbital standard Young tableau** of type  $[\vec{\lambda}]$  is an orbital tableau of type  $[\vec{\lambda}]$  such that  $n \in T^0 \cup \dots \cup T^{d-1}$ . We denote by  $\text{OSYT}_n[\vec{\lambda}]$  the set of all  $n$ -orbital  $r$ -tableaux of type  $\vec{\lambda}$ .

More precisely, let  $T$  be a standard Young  $r$ -tableau of shape  $\vec{\lambda}$ . From (15), it follows that all the possible orbital tableaux of type  $[\vec{\lambda}]$ , have shapes obtained from that of  $\vec{\lambda}$  by applying  $i \cdot d$ -times the shift operator (13), for  $i = 0, \dots, p-1$ .

*Example 7.1:* Let  $r = 6$  and  $n = 17$ . If  $p = 3$  and so  $d = 2$ , then the two tableaux  $T_1$  and  $T_2$  in Figure 1 are of the same type

$$[\vec{\lambda}] = [(1), (2), (2, 1), (1, 1), (3, 1), (2)];$$

$T_1$  is  $n$ -orbital, while  $T_2$  is not. Differently, for  $p = 2$ , and  $d = 3$  the two tableaux  $T_1$  and  $T_2$  are not in the same orbit  $[\vec{\lambda}]$ . Nevertheless,  $T_2$  is an  $n$ -orbital tableau of type  $[(3, 1), (2), (1), (2), (2, 1), (1, 1)]$ .

## 8. Colored-descent representations

The module of coinvariants  $\mathbb{C}[\mathbf{x}]_H$  has a natural grading induced from that of  $\mathbb{C}[\mathbf{x}]$ . If we denote by  $R_k$  its  $k^{\text{th}}$  homogeneous component, we have

$$\mathbb{C}[\mathbf{x}]_H = \bigoplus_{k \geq 0} R_k.$$

Since the action (9) preserves the degree, every homogeneous component  $R_k$  is itself an  $H$ -module. In this section we introduce a set of  $H$ -modules  $R_{\mathcal{D}, \mathcal{C}}$  which decompose  $R_k$ . The representations  $R_{\mathcal{D}, \mathcal{C}}$ , called **colored-descent representations**, generalize to all groups  $G(r, p, n)$ , the descent representations introduced for  $S_n$  and  $B_n$  by Adin, Brenti and Roichman in [3], see also [8] for the case of  $D_n$ . In the case of  $G(r, n)$ , a Solomon's descent algebra approach to these representations has been done by Baumann and Hohlweg [6]. Since their study is restricted to wreath products, it will be interesting to extend their results to all complex reflection groups, thus getting characters of all our modules as images of a particular class of elements of the group algebra.

We use most of the notion and tools introduced in Section 4. In particular, let us see the first important feature of the colored-descent basis elements.

**PROPOSITION 8.1:** *Let  $\gamma \in \Gamma$ , and  $h \in H$ . Then*

$$h \cdot \mathbf{x}_\gamma = \sum_{\{u \in \Gamma: \lambda(\mathbf{x}_u) \trianglelefteq \lambda(\mathbf{x}_\gamma)\}} n_u \mathbf{x}_u + y,$$

where  $c_u \in \mathbb{C}$ , and  $y \in \mathcal{I}_H$ .

*Proof:* Expand  $M = h \cdot \mathbf{x}_\gamma$  as in (12). Note that  $f_{\mu(M')} \notin \mathcal{I}_H$  if and only if  $\mu(M') = \emptyset$ . Suppose that  $M'$  gives a nonzero contribution to this expansion of  $M$ . This implies  $M' = \mathbf{x}_{\gamma(M')}$ . If we let  $u = \gamma(M')$ , then  $\lambda(\mathbf{x}_u) = \lambda(M') \trianglelefteq \lambda(M) = \lambda(h \cdot \mathbf{x}_\gamma) = \lambda(\mathbf{x}_\gamma)$ . ■

If  $|\lambda| = k$  then by Proposition 8.1:

$$J_\lambda^\trianglelefteq := \text{span}_{\mathbb{C}}\{\mathbf{x}_\gamma + \mathcal{I}_H \mid \gamma \in \Gamma, \lambda(\mathbf{x}_\gamma) \trianglelefteq \lambda\} \text{ and}$$

$$J_\lambda^\triangleleft := \text{span}_{\mathbb{C}}\{\mathbf{x}_\gamma + \mathcal{I}_H \mid \gamma \in \Gamma, \lambda(\mathbf{x}_\gamma) \triangleleft \lambda\}$$

are submodules of  $R_k$ . Their quotient is still an  $H$ -module, denoted by  $R_\lambda := J_\lambda^\trianglelefteq / J_\lambda^\triangleleft$ .

For any  $\mathcal{D} \subseteq [n]$  we define the partition  $\lambda_{\mathcal{D}} := (\lambda_1, \dots, \lambda_n)$ , where  $\lambda_i := |\mathcal{D} \cap [i, n]|$ . For  $\mathcal{D} \subseteq [n-1]$  and  $\mathcal{C} \in [0, r-1]^{n-1} \times [d]$ , we define the vector

$$\lambda_{\mathcal{D}, \mathcal{C}} := r \cdot \lambda_{\mathcal{D}} + \mathcal{C},$$

where the sum stands for sum of vectors. The following two observations are important in our analysis.

**Remark 8.2:** For any  $\gamma, \gamma' \in \Gamma$  we have:

- 1)  $\lambda(\mathbf{x}_\gamma) = \lambda_{\mathcal{D}, \mathcal{C}}$ , where  $\mathcal{D} = \text{Des}(\gamma)$  and  $\mathcal{C} = \text{Col}(\gamma)$
- 2)  $\lambda(\mathbf{x}_\gamma) = \lambda(\mathbf{x}_{\gamma'})$  if and only if  $\text{Des}(\gamma) = \text{Des}(\gamma')$  and  $\text{Col}(\gamma) = \text{Col}(\gamma')$ .

From now on we denote

$$R_{\mathcal{D}, \mathcal{C}} := R_{\lambda_{\mathcal{D}, \mathcal{C}}},$$

and by  $\bar{\mathbf{x}}_\gamma$  the image of the colored-descent basis element  $\mathbf{x}_\gamma \in J_{\lambda_{\mathcal{D}, \mathcal{C}}}^\triangleleft$  in the quotient  $R_{\mathcal{D}, \mathcal{C}}$ . The next proposition follows from the definition of  $R_{\mathcal{D}, \mathcal{C}}$ , and by Remark 8.2.

**PROPOSITION 8.3:** For any  $\mathcal{D} \subseteq [n-1]$  and  $\mathcal{C} \in [0, r-1]^{n-1} \times [d]$ , the set

$$\{\bar{\mathbf{x}}_\gamma : \gamma \in \Gamma, \text{Des}(\gamma) = \mathcal{D} \text{ and } \text{Col}(\gamma) = \mathcal{C}\}$$

is a basis of  $R_{\mathcal{D}, \mathcal{C}}$ .

The  $H$ -modules  $R_{\mathcal{D}, \mathcal{C}}$  are called **colored-descent representations**. The dimension of  $R_{\mathcal{D}, \mathcal{C}}$ , is then given by the number of elements in  $\Gamma$  with descent set  $\mathcal{D}$  and color vector  $\mathcal{C}$ . They decompose the  $k^{\text{th}}$  component of  $\mathbb{C}[\mathbf{x}]_H$  as follows.

**THEOREM 8.4:** For every  $0 \leq k \leq r \binom{n}{2} + n(d-1)$ ,

$$(17) \quad R_k \cong \bigoplus_{\mathcal{D}, \mathcal{C}} R_{\mathcal{D}, \mathcal{C}}$$

as  $H$ -modules, where the sum is over all  $\mathcal{D} \subseteq [n-1], \mathcal{C} \in [0, r-1]^{n-1} \times [d]$  such that

$$r \cdot \sum_{i \in \mathcal{D}} i + \sum_{j \in \mathcal{C}} j = k.$$

*Proof:* By Theorem 4.3, the set  $\{\mathbf{x}_\gamma + \mathcal{I}_H : \text{fmaj}(\gamma) = k\}$  is a basis for  $R_k$ , and so the isomorphism (17) as  $\mathbb{C}$ -vector spaces is clear. Now, let  $\lambda^{(1)} < \lambda^{(2)} < \dots < \lambda^{(t)}$  be a linear extension of the dominance order on the set  $\{\lambda(\mathbf{x}_\gamma) : \text{fmaj}(\gamma) = k\}$ , i.e.,  $\lambda^{(i)} \triangleleft \lambda^{(j)}$  implies  $i \leq j$ . Consider the following flag of subspaces of  $R_k$ ,  $M_i := \text{span}_{\mathbb{C}}\{\mathbf{x}_\gamma : \lambda(\mathbf{x}_\gamma) \leq \lambda^{(i)}\}$  for  $i \in [r]$ , and  $M_0 := 0$ . By Proposition 8.1,  $M_i$  is an  $H$ -submodule of  $R_k$ , and it is clear that  $M_i/M_{i-1} \cong R_{\lambda^{(i)}}$  for  $i \in [r]$ . By Maschke's theorem we can easily prove, by induction on  $n-i$ , that

$$R_k \cong M_i \oplus R_{\lambda^{(i+1)}} \oplus R_{\lambda^{(i+2)}} \oplus \dots \oplus R_{\lambda^{(t)}}$$

for all  $i \in [0, t]$ . The claim follows for  $i = 0$ , since  $M_0 = 0$ . ■

### 9. Stanley's generalized formula

In this section we prove a generalization of Stanley's Formula [28, Theorem 7.21.2], and a technical lemma that will be fundamental in the proof of Theorem 10.5. We need to introduce some more notation.

A **reverse semi-standard Young tableau of shape**  $\lambda$  is obtained by inserting positive integers into the diagram of  $\lambda$  in such a way that the entries decrease weakly along rows and strictly down columns. The set of all such tableaux is denoted by  $\text{RSSYT}(\lambda)$ . This notion generalizes to  $r$ -partitions as follows: Let  $\vec{\lambda} = (\lambda^0, \dots, \lambda^{r-1}) \in \mathcal{P}_{r,n}$ . A **reverse semi-standard Young tableau of shape**  $\vec{\lambda}$  is an  $r$ -tuple of reverse semi-standard Young tableaux  $\mathcal{T} = (\mathcal{T}^0, \dots, \mathcal{T}^{r-1})$ , where each  $\mathcal{T}^i$  has shape  $\lambda^i$ , and every entry in  $\mathcal{T}^i$  is congruent to  $i + 1 \pmod{r}$ , for all  $0 \leq i \leq r - 1$ . The set of all such tableaux is denoted by  $\text{RSSYT}(\vec{\lambda})$ . To every  $\mathcal{T} \in \text{RSSYT}(\vec{\lambda})$ , we associate the **entries partition** of  $\mathcal{T}$ :

$$(18) \quad \theta(\mathcal{T}) := (\theta_1, \dots, \theta_n),$$

made of the entries of  $\mathcal{T}$  in weakly decreasing order.

Similarly to [3, Section 5], for every  $\vec{\lambda} \in \mathcal{P}_{r,n}$  we define a map

$$\begin{aligned} \phi_{\vec{\lambda}} : \text{RSSYT}(\vec{\lambda}) &\rightarrow \text{SYT}(\vec{\lambda}) \times \mathbb{N}^n \\ \phi_{\vec{\lambda}}(\mathcal{T}) &\mapsto (T, \Delta), \end{aligned}$$

as follows:

$T$ ) Let  $\theta(\mathcal{T}) = (\theta_1, \dots, \theta_n)$  be the entries partition (18) of  $\mathcal{T}$ . Then  $T$  is the standard Young tableau of the same shape  $\vec{\lambda}$  of  $\mathcal{T}$ , with entry  $i$  in the same cell in which  $\mathcal{T}$  has  $\theta_i$ , for all  $i \in [n]$ . If some of the entries of  $\mathcal{T}$  are equal then they are in different columns and the corresponding entries of  $T$  will be chosen, increasing from left to right.

$\Delta$ ) For every  $i \in [n]$  let

$$\Delta_i := \frac{\theta_i - f_i(T) - \theta_{i+1} + f_{i+1}(T)}{r},$$

and set  $\theta_{n+1} = 1$ .

*Example 9.1:* Let  $r = 3$ ,  $n = 11$  and let  $\mathcal{T} = (\mathcal{T}^0, \mathcal{T}^1, \mathcal{T}^2)$  be the reverse standard Young 3-tableau of shape  $\vec{\lambda} = ((1, 1), (2, 2), (3, 1, 1))$  in Figure 2. Note that each entry in  $\mathcal{T}_i$  is congruent to  $i+1 \pmod{3}$ . The entries partition of  $\mathcal{T}$  is  $\theta(\mathcal{T}) = (14, 14, 10, 9, 9, 8, 7, 6, 5, 3, 3)$ . Let us compute the image of  $\mathcal{T}$ ,  $\phi_{\vec{\lambda}}(\mathcal{T}) = (T, \Delta)$ . The standard Young tableau  $T$  is drawn in Figure 2. Now,  $\text{Des}(T) = \{3, 7, 9\}$



and  $f(T) = (10, 10, 9, 8, 8, 7, 6, 5, 4, 2, 2)$ . It follows that  $(\theta(T) - f(T))_{i=1}^n = (4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ , and so  $\Delta = (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ .

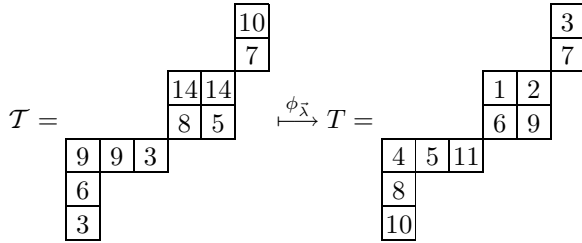


Figure 2. The bijection  $\phi_{\vec{\lambda}}$

The following remark is not hard to see.

*Remark 9.2:* For every  $\vec{\lambda} \in \mathcal{P}_{r,n}$ , let  $\phi_{\vec{\lambda}}(T) = (T, \Delta)$ . Then

- 1) the map  $\phi_{\vec{\lambda}}$  is a bijection;
- 2)  $\theta_i - 1 = f_i(T) + \sum_{j \geq i} r \Delta_j$  for all  $i \in [n]$ .

Let  $\mathbf{y} = y_1, y_2, \dots$  be an infinite set of variables. Define  $\mathbb{C}[[y_1, y_1 y_2, \dots]]$  to be the ring of formal power series in the countable number of variables  $y_1, y_1 y_2, \dots, y_1 y_2 \cdots y_k, \dots$ . A linear basis for  $\mathbb{C}[[y_1, y_1 y_2, \dots]]$  consists of the monomials  $\mathbf{y}^\lambda := y_1^{\lambda_1} \cdots y_n^{\lambda_n}$  for all partitions  $\lambda = (\lambda_1, \dots, \lambda_n)$ .

Let  $\iota: \mathbb{C}[[y_1, y_1 y_2, \dots]] \rightarrow \mathbb{C}[[q_1, q_1 q_2, \dots]]$  be the map defined on the generators by

$$\iota(\mathbf{y}^\lambda) := \mathbf{q}^{\lambda'},$$

and extended by linearity. Here, as usual,  $\lambda'$  denotes the conjugate partition of  $\lambda$ . Note that  $\iota$  is not a ring homomorphism.

The following proposition is a generalization of Lemma 5.10 of [3], which itself is a “two dimensional” generalization of Stanley’s Formula [28, Theorem 7.21.2].

**PROPOSITION 9.3** (Stanley’s generalized formula): *Let  $\vec{\lambda} \in \mathcal{P}_{r,n}$ . Then*

$$\iota[s_{\lambda^0}(\mathbf{z}^0) \cdot s_{\lambda^1}(\mathbf{z}^1) \cdots s_{\lambda^{r-1}}(\mathbf{z}^{r-1})] = \sum_{T \in \text{SYT}(\vec{\lambda})} \prod_{i=1}^n q_i^{f_i(T)} \frac{1}{\prod_{i=1}^n (1 - q_1^r q_2^r \cdots q_i^r)},$$

where

$$\begin{aligned} \mathbf{z}^0 &:= (1, y_1 y_2 \cdots y_r, y_1 \cdots y_{2r}, \dots) \\ \mathbf{z}^1 &:= (y_1, y_1 y_2 \cdots y_{r+1}, y_1 y_2 \cdots y_{2r+1}, \dots) \\ &\vdots \\ \mathbf{z}^{r-1} &:= (y_1 y_2 \cdots y_{r-1}, y_1 y_2 \cdots y_{2r-1}, \dots). \end{aligned} \tag{19}$$

*Proof:* Let  $\lambda$  be a partition of  $n$ . By [28, Proposition 7.10.4], the Schur function  $s_\lambda$  can be written as

$$s_\lambda(\mathbf{y}) = \sum_{\mathcal{T} \in \text{RSSYT}(\lambda)} \prod_{i=1}^{\infty} y_i^{m_i(\mathcal{T})},$$

where  $m_i(\mathcal{T})$  is the number of cells of  $\mathcal{T}$  with entry  $i$ . Let  $\mathcal{T} = (\mathcal{T}^0, \dots, \mathcal{T}^{r-1})$  be a reverse semi-standard Young  $r$ -tableaux of shape  $\vec{\lambda}$ . Note that for every  $i = 0, \dots, r-1$ , the tableau  $\frac{\mathcal{T}^i - (i+1)}{r}$  belongs to  $\text{RSSYT}(\lambda^i)$ , where  $\frac{\mathcal{T}^i - (i+1)}{r}$  is obtained by subtracting  $i+1$  and then dividing by  $r$  every entry of  $\mathcal{T}^i$ .

Now, let  $\mathbf{y}^j = y_1^j, y_2^j, \dots$ , for  $0 \leq j \leq r-1$ , be a set of  $r$  sequences of independent variables. From the previous observation, it follows that

$$(20) \quad s_{\lambda^0}(\mathbf{y}^0) s_{\lambda^1}(\mathbf{y}^1) \cdots s_{\lambda^{r-1}}(\mathbf{y}^{r-1}) = \sum_{\mathcal{T}} \prod_{i=1}^{\infty} (y_i^0)^{m_{ri-r+1}(\mathcal{T}^0)} (y_i^1)^{m_{ri-r+2}(\mathcal{T}^1)} \cdots (y_i^{r-1})^{m_{ri}(\mathcal{T}^{r-1})},$$

where  $\mathcal{T}$  runs over all  $\text{RSSYT}(\vec{\lambda})$ . Now, if we plug into (20) the following substitutions:

$$\begin{array}{lll} y_1^0 = 1 & y_2^0 = y_1 \cdots y_r & y_3^0 = y_1 \cdots y_{2r} \quad \cdots \\ y_1^1 = y_1 & y_2^1 = y_1 \cdots y_{r+1} & y_3^1 = y_1 \cdots y_{2r+1} \quad \cdots \\ \vdots & \vdots & \vdots \\ y_1^{r-1} = y_1 \cdots y_{r-1} & y_2^{r-1} = y_1 \cdots y_{2r-1} & y_3^{r-1} = y_1 \cdots y_{3r-1} \quad \cdots \end{array}$$

we get

$$s_{\lambda^0}(\mathbf{z}^0) s_{\lambda^1}(\mathbf{z}^1) \cdots s_{\lambda^{r-1}}(\mathbf{z}^{r-1}) = \sum_{\mathcal{T} \in \text{RSSYT}(\vec{\lambda})} \prod_{i=1}^{\infty} y_i^{m_{>i}(\mathcal{T})},$$

where  $m_{>i}(\mathcal{T})$  is the number of cells of  $\mathcal{T}$  with entry strictly bigger than  $i$ , and  $\mathbf{z}^i$  are defined as in (19). Let  $\mu(\mathcal{T})$  be the partition  $(m_{>1}(\mathcal{T}), m_{>2}(\mathcal{T}), \dots)$ . Clearly  $\mu_1(\mathcal{T}) \leq n$ . It is not hard to see that the conjugate partition of  $\mu(\mathcal{T})$  is the entries partition of  $\mathcal{T}$ , with entries lowered by 1, i.e.,

$$\mu(\mathcal{T})' = (\theta_1 - 1, \dots, \theta_n - 1).$$

Now the result follows by Remark 9.2.

$$\begin{aligned} \iota[s_{\lambda^0}(\mathbf{z}^0) s_{\lambda^1}(\mathbf{z}^1) \cdots s_{\lambda^{r-1}}(\mathbf{z}^{r-1})] &= \sum_{\mathcal{T} \in \text{RSSYT}(\vec{\lambda})} \prod_{i=1}^n q_i^{\theta_i-1} \\ &= \sum_{\substack{\mathcal{T} \in \text{SYT}(\vec{\lambda}) \\ (\Delta_1, \dots, \Delta_n) \in \mathbb{N}^n}} \prod_{i=1}^n q_i^{f_i(\mathcal{T})} \prod_{j=1}^n (q_1^r \cdots q_j^r)^{\Delta_j}. \quad \blacksquare \end{aligned}$$

LEMMA 9.4: Let  $T = (T^0, \dots, T^{r-1})$  be a Young tableau of total size  $n$  such that  $n \notin T^{r-1}$  and let  $T^\circ = (T^{r-1}, T^0, \dots, T^{r-2})$  be the Young tableau obtained from  $T$  by a 1-shift: Then for all  $i = 1, \dots, n$

$$f_i(T^\circ) = f_i(T) + 1.$$

*Proof:* We prove by backward induction on  $n$ . For  $i = n$ , since  $n \notin T^{r-1}$ , we clearly have  $f_n(T^\circ) = f_n(T) + 1$ . Hence let  $i < n$ . If  $i$  and  $i + 1$  are both in the same component, then we have  $c_i(T^\circ) - c_i(T) = c_{i+1}(T^\circ) - c_{i+1}(T)$ . Now, if  $i \in \text{Des}(T)$  then also  $i \in \text{Des}(T^\circ)$  thus  $d_i(T^\circ) - d_i(T) = d_{i+1}(T^\circ) - d_{i+1}(T)$ , and we conclude  $f_i(T^\circ) - f_i(T) = f_{i+1}(T^\circ) - f_{i+1}(T) = 1$  by the induction hypothesis. If  $i \notin \text{Des}(T)$  then the argument is similar.

We turn now to the more interesting case, where  $i, i + 1$  are not in the same component. We assume first that  $i, i + 1 \notin T^{r-1}$ . In this case, we clearly have  $c_i(T^\circ) - c_i(T) = c_{i+1}(T^\circ) - c_{i+1}(T)$ . Since  $i, i + 1 \notin T^{r-1}$ ,  $i \in \text{Des}(T)$  if and only if  $i \in \text{Des}(T^\circ)$ , and thus we have  $d_i(T^\circ) - d_i(T) = d_{i+1}(T^\circ) - d_{i+1}(T)$ . Hence, the result follows by the induction hypothesis as above.

If  $i \in T^{r-1}$  but  $i + 1 \notin T^{r-1}$  then  $i \notin \text{Des}(T)$  but  $i \in \text{Des}(T^\circ)$  so we have  $d_i(T^\circ) - d_i(T) = d_{i+1}(T^\circ) + 1 - d_{i+1}(T)$ . We also have  $c_i(T^\circ) - c_i(T) = 1 - r$  and  $c_{i+1}(T^\circ) - c_{i+1}(T) = 1$ . Thus  $c_i(T^\circ) - c_i(T) = c_{i+1}(T^\circ) - c_{i+1}(T) - r$ . Finally we have:

$$\begin{aligned} f_i(T^\circ) - f_i(T) &= r \cdot (d_i(T^\circ) - d_i(T)) + c_i(T^\circ) - c_i(T) \\ &= r \cdot (d_{i+1}(T^\circ) + 1 - d_{i+1}(T)) + c_{i+1}(T^\circ) - c_{i+1}(T) - r \\ &= f_{i+1}(T^\circ) - f_{i+1}(T) = 1 \end{aligned}$$

by the induction hypothesis. The case  $i \in T^r, i + 1 \notin T^r$  is similar. ■

## 10. Decomposition of $R_{\mathcal{D}, \mathcal{C}}$

In this section we prove a simple combinatorial description of the multiplicities of the irreducible representations of  $H$  in  $R_{\mathcal{D}, \mathcal{C}}$ .

For an element  $h \in H$  let the (graded) trace of its action on the polynomial ring  $\mathbb{C}[\mathbf{x}]$  be

$$(21) \quad \text{Tr}_{\mathbb{C}[\mathbf{x}]}(h) := \sum_M \langle h \cdot M, M \rangle \cdot \mathbf{q}^{\lambda(M)},$$

where the sum is over all monomials  $M \in \mathbb{C}[\mathbf{x}]$ ,  $\lambda(M)$  is the exponent partition of  $M$ , and the inner product is such that the set of all monomials is an orthonormal basis for  $\mathbb{C}[\mathbf{x}]$ . Note that  $\langle h \cdot M, M \rangle \in \{0, \zeta^i : i = 0, \dots, r - 1\}$ .

LEMMA 10.1: *Let  $h \in H$  be of cycle type  $\vec{\alpha} = (\alpha^0, \dots, \alpha^{r-1})$ . Then the graded trace of its action on  $\mathbb{C}[\mathbf{x}]$  is*

$$\mathrm{Tr}_{\mathbb{C}[\mathbf{x}]}(h) = \iota[p_{\vec{\alpha}}(\mathbf{z}^0, \mathbf{z}^1, \dots, \mathbf{z}^{r-1})],$$

where the parameters  $\mathbf{z}^k$  are defined as in (18).

*Proof:* Let  $h \in H$  be of cycle type  $(\alpha^0, \dots, \alpha^{r-1})$ . Decompose  $h$  into cycles according to the cycle type of  $h$ , as explained in Section 5, to get  $h = \mathfrak{h}^0 \cdots \mathfrak{h}^{r-1}$ , in such a way that  $\mathfrak{h}^j$  is a product of cycles of type  $j$ , for each  $0 \leq j \leq r-1$ . We further decompose each  $\mathfrak{h}^j$  into disjoint cycles  $\mathfrak{h}^{j,1}, \dots, \mathfrak{h}^{j,\ell(\alpha^j)}$ . For example, as we observed in Section 5,  $h = 62^5 4^4 3^1 1^6 5^3 \in G(8, 2, 6)$  can be decomposed as a product of  $\mathfrak{h}^{1,1} = (1^6 6 5^3)$ ,  $\mathfrak{h}^{5,1} = (3^1 4^4)$  and  $\mathfrak{h}^{5,2} = (2^5)$ .

Every monomial  $M \in \mathbb{C}[\mathbf{x}]$  satisfying  $\langle h \cdot M, M \rangle \neq 0$  is of the form  $M = M^0 \cdots M^{r-1}$ , where the variables of  $M^j$  have indices appearing in  $\mathfrak{h}^j$  and all have the same exponent. Once again, we can further decompose each  $M^j$  as  $M^j = M^{j,1} \cdots M^{j,\ell(\alpha^j)}$  according to the cycle decomposition of  $\mathfrak{h}^j$ . It is easy to see that  $h \cdot M = \prod_{j=0}^{r-1} \prod_{i=1}^{\ell(\alpha^j)} \mathfrak{h}^{j,i} \cdot M^{j,i}$ . So, given such an  $M$ , we can calculate the contribution of its parts to the trace separately. Fix  $0 \leq j \leq r-1$  and  $1 \leq i \leq \ell(\alpha^j)$ . To every  $e^{j,i} \in \mathbb{N}$  corresponds a unique monomial  $M^{j,i}$  with  $\langle \mathfrak{h}^{j,i} \cdot M^{j,i}, M^{j,i} \rangle \neq 0$  such that  $e^{j,i}$  is the common exponent for the factor  $M^{j,i}$ . Moreover  $\langle \mathfrak{h}^{j,i} \cdot M^{j,i}, M^{j,i} \rangle = \zeta^{j \cdot e^{j,i}}$ . On the other hand,  $\lambda(M^{j,i})$  consists of  $\alpha_i^j$  copies of  $e^{j,i}$ , and we have

$$\iota^{-1}[\mathbf{q}^{\lambda(M^{j,i})}] = \mathbf{y}^{\lambda(M^{j,i})'} = (y_1 \cdots y_{e^{j,i}})^{\alpha_i^j}.$$

The last expression is a summand in  $p_{\alpha_i^j}(\mathbf{z}^{e^{j,i}})$ . Here,  $e^{j,i}$  is taken in mod  $r$ , and  $\mathbf{z}^k$  is defined as in (18). Continuing our example, the monomial  $M = x_1^2 x_6^2 x_5^2 \cdot x_3^4 x_4^4 \cdot x_2^5$  is such that  $\langle h \cdot M, M \rangle \neq 0$ . For  $M^{5,1} = x_3^4 x_4^4$ , we have  $\langle \mathfrak{h}^{5,1} M^{5,1}, M^{5,1} \rangle = \zeta^4$ , and  $\iota^{-1}[\mathbf{q}^{\lambda(M^{5,1})}] = (y_1 \cdots y_4)^2$ , with  $(y_1 \cdots y_4)^2$  appearing as a summand in  $p_2(\mathbf{z}^4)$ .

By summing over all the possibilities for  $i, j$  and  $e^{j,i}$ , we obtain

$$\mathrm{Tr}_{\mathbb{C}[\mathbf{x}]}(h) = \iota \left( \prod_{j=0}^{r-1} \prod_{i=1}^{\ell(\alpha^j)} \sum_{k=0}^{r-1} \zeta^{j \cdot k} p_{\alpha_i^j}(\mathbf{z}^k) \right) = \iota[p_{\vec{\alpha}}(\mathbf{z}^0, \mathbf{z}^1, \dots, \mathbf{z}^{r-1})]. \quad \blacksquare$$

Recall the notation introduced in Section 4. Clearly, every monomial  $N \in \mathbb{C}[\mathbf{x}]$  can be written as

$$(22) \quad N = \vartheta_n^t \cdot M,$$

with  $M \in S := \mathbb{C}[\mathbf{x}]/(\vartheta_n)$ , and  $t \in \mathbb{N}$ .

From the expansion (12), it follows that the set  $\{\vartheta_n^t \vartheta_\mu \mathbf{x}_\gamma\}$  is an homogeneous basis for  $\mathbb{C}[\mathbf{x}]$ , where  $\gamma \in \Gamma$ ,  $\mu \in \mathcal{P}(n-1)$  and  $t \in \mathbb{N}$ . Here  $\mathcal{P}(n-1)$  denotes the set of all partitions  $\mu$  with largest part at most  $n-1$ .

We associate now to any monomial  $N = \vartheta_n^t \cdot M$  in  $\mathbb{C}[\mathbf{x}]$  a triple

$$(23) \quad \begin{aligned} \phi: \mathcal{M} &\longrightarrow \Gamma(r, p, n) \times \mathcal{P}(n-1) \times \mathbb{N} \\ N &\longmapsto (\gamma(M), \mu(M), t). \end{aligned}$$

Here  $\mathcal{M}$  denotes the set of all monomials in  $\mathbb{C}[\mathbf{x}]$ ,  $\gamma(M)$  and  $\mu(M)$  are the color index permutation and the complementary partition of the monomial  $M \in S$  defined in (22).

LEMMA 10.2: *The map  $\phi$  is a bijection*

*Proof:* The map is clearly into  $\Gamma \times \mathcal{P}(n-1) \times \mathbb{N}$  since  $\mu'(M)$  has at most  $n-1$  parts. On the other hand, to each triple  $(\gamma, \mu, t) \in \Gamma \times \mathcal{P}(n-1) \times \mathbb{N}$  we associate the monomial  $N = \vartheta_n^t \cdot M$ , where  $M$  is the  $\sqsubset$ -maximal monomial in the expansion of  $\vartheta_\mu \mathbf{x}_\gamma$ . Clearly  $\phi(N) = (\gamma, \mu, t)$ . ■

*Example 10.3:* Consider the monomial  $N = x_1^{17} x_2^{14} x_3^7$ . Then  $N = \vartheta_3^2 \cdot M$ , where  $M = x_1^{11} x_2^8 x_3$  is the monomial that appeared in Example 4.2. It follows that  $\gamma(M) = 1^5 2^2 3^1 \in \Gamma(6, 2, 3)$ , and  $\mu(M) = (2)$ . Hence  $N \mapsto (1^5 2^2 3, (2), 2)$ .

Since  $\lambda(N) = \lambda(M) + (dt)^n$  and  $\lambda(M) = \lambda(\mathbf{x}_{\gamma(M)}) + r \cdot \mu'(M)$ , we get

$$(24) \quad \lambda(N) = \lambda(\mathbf{x}_\gamma) + r \cdot \mu'(M) + (dt)^n.$$

Similarly to (21), we let

$$(25) \quad \text{Tr}_{\mathbb{C}[\mathbf{x}]_H}(h) := \sum_{\gamma \in \Gamma} \langle h \cdot (\mathbf{x}_\gamma + \mathcal{I}_H), \mathbf{x}_\gamma + \mathcal{I}_H \rangle \cdot \mathbf{q}^{\lambda(\mathbf{x}_\gamma)},$$

where the inner product is such that the colored-descent basis is orthonormal. The following result relates the two traces.

LEMMA 10.4: *Let  $n \in \mathbb{C}[\mathbf{x}]$  and  $h \in H$ . Then*

$$\text{Tr}_{\mathbb{C}[\mathbf{x}]}(h) = \text{Tr}_{\mathbb{C}[\mathbf{x}]_H}(h) \cdot \frac{1}{1 - q_1^d \cdots q_n^d} \prod_{i=1}^{n-1} \frac{1}{1 - q_1^r \cdots q_i^r}.$$

*Proof:* We compute the trace (21) with respect to the homogeneous basis  $\{\vartheta_n^t \vartheta_\mu \mathbf{x}_\gamma\}$ , (where  $t \in \mathbb{N}$ ,  $\mu \in \mathcal{P}(n-1)$  and  $\gamma \in \Gamma$ ), using the inner product

which makes this basis orthonormal. Note that by Proposition 8.1, the action of  $h$  on  $\vartheta_n^t \vartheta_\mu \mathbf{x}_\gamma$  is triangular and thus the trace (21) now looks like

$$\mathrm{Tr}_{\mathbb{C}[\mathbf{x}]}(h) = \sum_{\substack{\mu \in \mathcal{P}(n-1) \\ \gamma \in \Gamma, t \in \mathbb{N}}} \langle h \cdot \vartheta_n^t \vartheta_\mu \mathbf{x}_\gamma, \vartheta_n^t \vartheta_\mu \mathbf{x}_\gamma \rangle \mathbf{q}^{\lambda(N)}$$

where  $N$  is the  $\square$ -maximal monomial in the expansion of  $\vartheta_n^t \vartheta_\mu \mathbf{x}_\gamma$ .

On the other hand, since  $\vartheta_n^t \vartheta_\mu$  is  $H$ -invariant, we have  $\langle h \cdot \vartheta_n^t \vartheta_\mu \mathbf{x}_\gamma, \vartheta_n^t \vartheta_\mu \mathbf{x}_\gamma \rangle = \langle h \cdot \mathbf{x}_\gamma, \mathbf{x}_\gamma \rangle$ , and thus from (25) and (24) we get

$$\begin{aligned} \mathrm{Tr}_{\mathbb{C}[\mathbf{x}]}(h) &= \mathrm{Tr}_{\mathbb{C}[\mathbf{x}]_H}(h) \cdot \sum_{\mu \in \mathcal{P}(n-1)} \mathbf{q}^{r \cdot \mu'} \cdot \sum_{t \in \mathbb{N}} \mathbf{q}^{(dt)^n} \\ &= \mathrm{Tr}_{\mathbb{C}[\mathbf{x}]_H}(h) \cdot \sum_{\ell(\lambda) \leq n-1} (q_1 \cdots q_n)^{r \cdot \lambda} \cdot \sum_{t \geq 0} (q_1^d \cdots q_n^d)^t. \quad \blacksquare \end{aligned}$$

**THEOREM 10.5:** *For every  $\mathcal{D} \subseteq [n-1]$  and  $\mathcal{C} \subseteq [0, r-1]^{n-1} \times [d]$ ,  $\vec{\lambda} \in \mathcal{P}_{r,n}$  and  $\delta \in C_{\vec{\lambda}}$ , the multiplicity of the irreducible representation of  $G(r, p, n)$  corresponding to the pair  $([\vec{\lambda}], \delta)$  in  $R_{\mathcal{D}, \mathcal{C}}$  is*

$$|\{T \in \mathrm{OSYT}_n[\vec{\lambda}] \mid \mathrm{Des}(T) = \mathcal{D}, \mathrm{Col}(T) = \mathcal{C}\}|.$$

*Proof:* Let  $h \in H$ . If  $h$  is of type  $\vec{\alpha} = (\alpha^0, \dots, \alpha^{r-1})$  then by Lemma 10.1 we have:

$$\mathrm{Tr}_{\mathbb{C}[\mathbf{x}]}(h) = \iota[p_{\vec{\alpha}}(\mathbf{z}^0, \dots, \mathbf{z}^{r-1})].$$

By the Frobenius Formula for  $G(r, p, n)$ , (Theorem 6.2), we have

$$\iota[p_{\vec{\alpha}}(\mathbf{z}^0, \dots, \mathbf{z}^{r-1})] = \iota \left[ \sum_{[\vec{\lambda}]} (\chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_1)} + \cdots + \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_{u(\vec{\lambda})})}) \cdot \sum_{\vec{\mu} \in [\vec{\lambda}]} s_{\mu^0}(\mathbf{z}^0) \cdots s_{\mu^{r-1}}(\mathbf{z}^{r-1}) \right]$$

where  $u(\vec{\lambda}) = |C_{\vec{\lambda}}|$ ,  $[\vec{\lambda}]$  runs over the orbits of the irreducible  $G$ -modules, and  $\vec{\mu}$  runs over the elements of the orbit of  $\vec{\lambda}$ . Let us fix an irreducible  $G$ -module indexed by  $\vec{\lambda}$  and consider the expression  $\sum_{\vec{\mu} \in [\vec{\lambda}]} s_{\mu^0}(\mathbf{z}^0) \cdots s_{\mu^{r-1}}(\mathbf{z}^{r-1})$ . If we denote by  $\vec{\mu}_0 = (\mu_0^0, \dots, \mu_0^r)$  one of the elements of  $[\vec{\lambda}]$  and by  $\delta_0^d$  the generator of  $G/H$ , then we can write  $[\vec{\lambda}] = \{(\vec{\mu}_0)^{\odot di} \mid 0 \leq i \leq p-1\}$ . By Lemma 9.3 we have

(26)

$$\iota \left[ \sum_{\vec{\mu} \in [\vec{\lambda}]} s_{\mu^0}(\mathbf{z}^0) \cdots s_{\mu^{r-1}}(\mathbf{z}^{r-1}) \right] = \sum_{\vec{\mu} \in [\vec{\lambda}]} \sum_{T \in \mathrm{SYT}(\vec{\mu})} \prod_{i=1}^n q_i^{f_i(T)} \frac{1}{\prod_{i=1}^n (1 - q_1^r q_2^r \cdots q_i^r)}.$$

The action of  $d$ -shifts gives us a bijection between the sets

$$\{T \in \text{SYT}(\vec{\mu}_0) \mid n \in T^0 \cup \dots \cup T^{d-1}\}$$

and

$$\{T \in \text{SYT}((\vec{\mu}_0)^{\odot d}) \mid n \in T^d \cup \dots \cup T^{2d-1}\}.$$

By iterating this procedure we obtain that each set of the form

$$\{T \in \text{SYT}((\vec{\mu}_0)^{\odot di}) \mid n \in T^{id} \cup \dots \cup T^{(i+1)d-1}\}$$

for  $1 \leq i \leq p-1$  is in a bijective correspondence with one of the sets  $\{T \in \text{SYT}((\vec{\mu}_0)^{\odot dj}) \mid n \in T^0 \cup \dots \cup T^{d-1}\}$  for some  $0 \leq j \leq p-1$ .

Before finishing the proof, let us consider a simple example. Let  $G = G(6, n)$  and  $H = G(6, 3, n)$ . Then the generator of  $G/H$  acts by a 2-shift. The bijection described above can be seen in the following diagram which has to be considered as lying in a torus screen.

$$\begin{array}{ccc}
 \{T \in S_{012345} \mid n \in T^0 \cup T^1\} & \{T \in S_{012345} \mid n \in T^2 \cup T^3\} & \{T \in S_{012345} \mid n \in T^4 \cup T^5\} \\
 \searrow & & \searrow \\
 \{T \in S_{450123} \mid n \in T^0 \cup T^1\} & \{T \in S_{450123} \mid n \in T^2 \cup T^3\} & \{T \in S_{450123} \mid n \in T^4 \cup T^5\} \\
 \searrow & & \searrow \\
 \{T \in S_{234501} \mid n \in T^0 \cup T^1\} & \{T \in S_{234501} \mid n \in T^2 \cup T^3\} & \{T \in S_{234501} \mid n \in T^4 \cup T^5\} \\
 \searrow & & \searrow \\
 \{T \in S_{012345} \mid n \in T^0 \cup T^1\} & \{T \in S_{012345} \mid n \in T^2 \cup T^3\} & \{T \in S_{012345} \mid n \in T^4 \cup T^5\} \\
 \searrow & & \searrow \\
 \{T \in S_{450123} \mid n \in T^0 \cup T^1\} & \{T \in S_{450123} \mid n \in T^2 \cup T^3\} & \{T \in S_{450123} \mid n \in T^4 \cup T^5\}
 \end{array}$$

where we write for example  $S_{012345}$  for  $\text{SYT}(\lambda^0, \dots, \lambda^5)$ .

By using Lemma 9.4 together with the above argument we obtain that the RHS of (26) is equal to

$$\sum_{\vec{\mu} \in [\vec{\lambda}]} \sum_{\substack{T \in \text{SYT}(\vec{\mu}) \\ n \in T^0 \cup \dots \cup T^{d-1}}} \prod_{i=1}^n q_i^{f_i(T)} \frac{\sum_{k=0}^{p-1} (q_1 \cdots q_n)^{kd}}{\prod_{i=1}^n (1 - q_1^r q_2^r \cdots q_i^r)}.$$

It follows that

$$\begin{aligned}
 \text{Tr}_{\mathbb{C}[\mathbf{x}]}(\gamma) &= \frac{1 - (q_1^d \cdots q_n^d)^p}{(1 - q_1^d \cdots q_n^d) \prod_{i=1}^n (1 - q_1^r q_2^r \cdots q_i^r)} \\
 &\quad \times \sum_{[\vec{\lambda}]} \left( \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_1)} + \cdots + \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_{u(\lambda)})} \right) \sum_{\vec{\mu} \in [\vec{\lambda}]} \sum_{\substack{T \in \text{SYT}(\vec{\mu}) \\ n \in T^0 \cup \dots \cup T^{d-1}}} \prod_{i=1}^n q_i^{f_i(T)}.
 \end{aligned}$$

Finally, from Lemma 10.4 we obtain

$$\begin{aligned} \mathrm{Tr}_{\mathbb{C}[\mathbf{x}]_H}(\gamma) &= \sum_{[\vec{\lambda}]} \left( \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_1)} + \cdots + \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_{u(\lambda)})} \right) \sum_{\vec{\mu} \in [\vec{\lambda}]} \sum_{\substack{T \in \mathrm{SYT}(\vec{\mu}) \\ n \in T^0 \cup \dots \cup T^{d-1}}} \prod_{i=1}^n q_i^{f_i(T)} \\ &= \sum_{[\vec{\lambda}]} \left( \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_1)} + \cdots + \chi_{\vec{\alpha}}^{([\vec{\lambda}], \delta_{u(\lambda)})} \right) \sum_{T \in \mathrm{OSYT}_n[\vec{\lambda}]} \prod_{i=1}^n q_i^{f_i(T)}. \end{aligned}$$

We conclude that the graded multiplicity of the irreducible  $H$ -module corresponding to the pair  $([\vec{\lambda}], \delta)$  in  $\mathbb{C}[\mathbf{x}]_H$  is

$$\sum_{T \in \mathrm{OSYT}_n[\vec{\lambda}]} \prod_{i=1}^n q_i^{f_i(T)} = \sum_{T \in \mathrm{OSYT}_n[\vec{\lambda}]} \mathbf{q}^{\lambda_{\mathrm{Des}(T), \mathrm{Col}(T)}}.$$

From the proof of Theorem 8.4, we obtain the decomposition

$$\mathbb{C}[\mathbf{x}]_H \simeq \bigoplus_{\mathcal{D}, \mathcal{C}} R_{\mathcal{D}, \mathcal{C}},$$

as graded  $H$ -modules. By Proposition 8.3 and the first part of Remark 8.2, it follows that  $R_{\mathcal{D}, \mathcal{C}}$  is the homogeneous component of multidegree  $\lambda_{\mathcal{D}, \mathcal{C}}$  in  $\mathbb{C}[\mathbf{x}]_H$ , and we are done. ■

As a consequence of Theorems 8.4 and 10.5, we obtain the following result that was first proved by Stembridge [31], using a different terminology.

**COROLLARY 10.6:** *For  $0 \leq k \leq r\binom{n}{2} + n(d-1)$ , the representation  $R_k$  is isomorphic to the direct sum  $\bigoplus m_{k, (\lambda, \delta)} V^{([\vec{\lambda}], \delta)}$ , where  $V^{([\vec{\lambda}], \delta)}$  is the irreducible representation of  $H$  labeled by  $([\vec{\lambda}], \delta)$ , and*

$$m_{k, ([\vec{\lambda}], \delta)} := |\{T \in \mathrm{OSYT}_n[\vec{\lambda}] \mid \mathrm{fmaj}(T) = k\}|.$$

## 11. Carlitz Identity

In the case of classical Weyl groups and wreath products, any major statistic is associated with a descent statistic and their joint distribution is given by a nice closed formula, called **Carlitz identity**. In this last section we show that this is the case also for the complex reflection groups  $G(r, p, n)$ .

For any partition  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}(n)$  we let, for every  $j \geq 0$ ,

$$m_j(\lambda) := |\{i \in [n] : \lambda_i = j\}|, \quad \text{and} \quad \binom{n}{m_0(\lambda), m_1(\lambda), \dots}$$

be the multinomial coefficient.



LEMMA 11.1: *Let  $n \in \mathbb{P}$ . Then*

$$(27) \quad \sum_{\ell(\lambda) \leq n} \binom{n}{m_0(\lambda), m_1(\lambda), \dots} \prod_{i=1}^n q_i^{\lambda_i} = \frac{\sum_{\gamma \in \Gamma} \prod_{i=1}^{n-1} q_i^{rd_i(\gamma) + c_i(\gamma)}}{(1 - q_1^d \cdots q_n^d) \prod_{i=1}^{n-1} (1 - q_1^r \cdots q_i^r)},$$

in  $\mathbb{C}[[q_1, \dots, q_n]]$ .

*Proof:* The LHS of the theorem is the multi graded Hilbert series of the polynomials ring  $\mathbb{C}[\mathbf{x}]$  by exponent partition. In fact,  $\binom{n}{m_0(\lambda), m_1(\lambda), \dots}$  is the number of monomials in  $\mathbb{C}[\mathbf{x}]$  with exponent partition equals to  $\lambda$ . On the other hand, by the bijection (23), we have

$$\begin{aligned} \sum_{N \in \mathbb{C}[\mathbf{x}]} \mathbf{q}^{\lambda(N)} &= \sum_{N = \vartheta_n^t M} \mathbf{q}^{\lambda(\mathbf{x}_{\gamma(M)}) + r \cdot \mu'(M) + (dt)^n} \\ &= \sum_{\gamma \in \Gamma} \mathbf{q}^{\lambda(\mathbf{x}_{\gamma})} \cdot \sum_{t \geq 0} (q_1^d \cdots q_n^d)^t \cdot \sum_{\mu \in \mathcal{P}(n-1)} q^{r\mu'} \\ &= \frac{\sum_{\gamma \in \Gamma} \prod_{i=1}^{n-1} q_i^{rd_i(\gamma) + c_i(\gamma)}}{(1 - q_1^d \cdots q_n^d) \prod_{i=1}^{n-1} (1 - q_1^r \cdots q_i^r)}. \quad \blacksquare \end{aligned}$$

Recall the definition of the flag-descent number given in (8).

THEOREM 11.2 (Carlitz identity for  $G(r, p, n)$ ): *Let  $n \in \mathbb{N}$ . Then*

$$(28) \quad \sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{h \in G(r, n, p)} t^{\text{fdes}(h)} q^{\text{fmaj}(h)}}{(1-t)(1-t^r q^r)(1-t^r q^{2r}) \cdots (1-t^r q^{(n-1)r})(1-t^d q^{nd})}.$$

*Proof:* The RHS is obtained by substituting  $q_1 = qt$ ,  $q_2 = \dots = q_n = q$  in (27). The identity from the LHS of (28) and the LHS of (27) is shown in [3, Corollary 6.4].  $\blacksquare$

We refer to Theorem 11.2 as the Carlitz identity for  $G(r, p, n)$ . It is worth noting that the powers of the  $q$ 's in the denominator,  $r, 2r, \dots, (n-1)r, nd$ , are actually the degrees of  $G(r, p, n)$ .

Differently, if we plug  $q_1 = \dots = q_n = q$  in (27), we get the following identity

$$\frac{1}{(1-q)^n} = \frac{\sum_{\gamma \in \Gamma} q^{\text{fmaj}(\gamma)}}{(1-q^{nd}) \prod_{i=1}^{n-1} (1-q^{ri})}.$$

Here, the LHS is the Hilbert series of the ring of polynomials (simply graded) while the RHS is the product of the Hilbert series of the module of coinvariants of  $G(r, p, n)$  by the Hilbert series of the invariants of  $G(r, p, n)$  (on the

denominator). Clearly, this equality reflects the isomorphism between graded  $H$ -modules

$$\mathbb{C}[\mathbf{x}] \simeq \mathbb{C}[\mathbf{x}]^H \otimes \mathbb{C}[\mathbf{x}]_H,$$

which holds if and only if  $H$  is a complex reflection groups.

## A. Appendix

In this section we present another proof for the Carlitz identity for  $G(r, p, n)$ , this time as a consequence of the Carlitz identity for  $G(r, n)$  which itself can be deduced either as a special case of Theorem 11.2 or by using a bijection analogues to (23). We start by presenting the Carlitz identity for  $G(r, n)$ .

**THEOREM A.1** (Carlitz identity for  $G$ ): *Let  $n \in \mathbb{N}$ . Then*

$$(29) \quad \sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{g \in G(r, n)} t^{\text{fdes}(g)} q^{\text{fmaj}(g)}}{(1-t)(1-t^r q^r)(1-t^r q^{2r}) \cdots (1-t^r q^{nr})}.$$

The proof we supply in this section points to an interesting connection between the group  $G(r, p, n)$  and its irreducible representations. Actually, the idea of the proof of Theorem 10.5 about the decomposition of colored descent representations into irreducibles, arises here in the course of the alternative proof of Theorem 11.2. More explicitly, the technical result we will use, Lemma A.2, carries the same role as Lemma 9.4 carries for standard Young tableaux. We start with some notations.

For every  $0 \leq i \leq r-1$ , we let

$$G_i = \{g = ((c_1, \dots, c_n), \sigma) \mid c_n = i\}.$$

Clearly we can decompose  $G$  and  $\Gamma$  as follows:

$$(30) \quad G = G_0 \uplus G_1 \uplus \cdots \uplus G_{r-1}, \text{ and}$$

$$(31) \quad \Gamma = G_0 \uplus G_1 \uplus \cdots \uplus G_{d-1},$$

where  $\uplus$  stands for disjoint union.

For  $g = ((c_1, \dots, c_n), \sigma) \in G$  and  $i \in \mathbb{N}$  define

$$g^i := ((\tilde{c}_1, \dots, \tilde{c}_n), \sigma),$$

where  $\tilde{c}_k \equiv c_k + i \pmod{r}$ .

Moreover, for any  $g = ((c_1, \dots, c_n), \sigma) \in G(r, n)$  we let

$$\hat{g} = ((c_1, \dots, c_{n-1}), \hat{\sigma}) \in G(r, n-1),$$

where for all  $i \in [n-1]$

$$\hat{\sigma}(i) := \begin{cases} \sigma(i) & \text{if } \sigma(i) < \sigma(n) \\ \sigma(i) - 1 & \text{if } \sigma(i) > \sigma(n). \end{cases}$$

For example, let  $g = ((4, 1, 3, 0, 2, 1), 416253) \in G(5, 6)$ . Then  $g^2 = ((1, 3, 0, 2, 4, 3), 416253) \in G(5, 6)$ , and  $\hat{g} = ((4, 1, 3, 0, 2), 31524) \in G(5, 5)$ . It is easy to see that

$$(32) \quad \text{Des}(\hat{g}) = \text{Des}(g) \cap [n-2] \quad \text{and} \quad \text{Col}(\hat{g}) = \text{Col}(g) \cap [n-1].$$

LEMMA A.2: *Let  $g \in G_0$ . Then for every  $i < r$*

$$(33) \quad \text{fdes}(g^i) = \text{fdes}(g) + i,$$

$$(34) \quad \text{fmaj}(g^i) = \text{fmaj}(g) + ni.$$

*Proof:* We proceed by induction on  $n$ . If  $n = 1$  then the statement is clearly true. Hence suppose  $n > 1$ , and let  $g = ((c_1, \dots, c_n), \sigma)$ . There are three cases to consider.

1)  $\sigma(n-1) > \sigma(n)$ .

In this case,  $c_{n-1} = c_n = 0$ , and so  $\hat{g} \in G_0(r, n-1)$ . From (32) we have

$$\text{Des}(g) = \text{Des}(\hat{g}) \cup \{n-1\} \quad \text{and} \quad \text{Des}(g^i) = \text{Des}(\hat{g}^i) \cup \{n-1\}.$$

Hence  $\text{des}(g) = \text{des}(\hat{g}) + 1$ ,  $\text{maj}(g) = \text{maj}(\hat{g}) + n - 1$ , and analogously  $\text{des}(g^i) = \text{des}(\hat{g}^i) + 1$ ,  $\text{maj}(g^i) = \text{maj}(\hat{g}^i) + n - 1$ . Since  $c_1(g) = c_1(\hat{g})$  and  $c_1(g^i) = c_1(\hat{g}^i)$ , by induction we get

$$\begin{aligned} \text{fdes}(g^i) &= r \cdot \text{des}(g^i) + c_1(g^i) = r \cdot \text{des}(\hat{g}^i) + r + c_1(\hat{g}^i) \\ &= \text{fdes}(\hat{g}^i) + r = \text{fdes}(\hat{g}) + i + r \\ &= \text{fdes}(g) + i. \end{aligned}$$

On the other hand, from (32) we get:  $\text{col}(g^i) = \text{col}(\hat{g}^i)$ . It follows that:

$$\begin{aligned} \text{fmaj}(g^i) &= r \cdot \text{maj}(\hat{g}^i) + rn - r + \text{col}(\hat{g}^i) \\ &= \text{fmaj}(\hat{g}^i) + rn - r = \text{fmaj}(\hat{g}) + in + rn - r \\ &= \text{fmaj}(g) + in. \end{aligned}$$

2)  $\sigma(n-1) < \sigma(n)$  and  $c_{n-1} = c_n$ .

In this case  $\text{Des}(g) = \text{Des}(\hat{g})$  and  $\text{Des}(g^i) = \text{Des}(\hat{g}^i)$ . Since  $\hat{g} \in G_0(r, n-1)$ , the result easily follows by induction just as in case 1).

3)  $\sigma(n-1) < \sigma(n)$  and  $c_{n-1} = k \neq 0$ .

Suppose first that  $i = r - k$ . In this case the position  $n - 1$ , which is not a descent for  $g$ , becomes a descent for  $g^i$ , hence  $\text{Des}(g^i) = \text{Des}(\hat{g}^i) \cup \{n - 1\}$ . It follows that:

$$(35) \quad \text{fdes}(g) = \text{fdes}(\hat{g}) \quad \text{and} \quad \text{fdes}(g^i) = \text{fdes}(\hat{g}^i) + r.$$

Now,  $c_{n-1}(\hat{g}^i) = c_{n-1} + r - k = 0$ , hence we can apply induction. Then for all  $j < r$  we have that  $\text{fdes}(\hat{g}^{r-k+j}) = \text{fdes}(\hat{g}^{r-k}) + j$ . In particular, if we set  $j = k$  then we obtain:

$$(36) \quad \text{fdes}(\hat{g}) = \text{fdes}(\hat{g}^{r-k+k}) = \text{fdes}(\hat{g}^{r-k}) + k.$$

Hence, from (35) and (36) we obtain:

$$\text{fdes}(g^{r-k}) = \text{fdes}(\hat{g}^{r-k}) + r = \text{fdes}(\hat{g}) + r - k = \text{fdes}(g) + r - k.$$

On the other hand it is clear that  $\text{maj}(g^{r-k}) = \text{maj}(\hat{g}^{r-k}) + n - 1$  and that  $\text{col}(g^{r-k}) = \text{col}(\hat{g}^{r-k}) + r - k$ . Hence

$$(37) \quad \begin{aligned} \text{fmaj}(g^{r-k}) &= r \cdot \text{maj}(g^{r-k}) + \text{col}(g^{r-k}) \\ &= r \cdot \text{maj}(\hat{g}^{r-k}) + rn - r + \text{col}(\hat{g}^{r-k}) + r - k \\ &= \text{fmaj}(\hat{g}^{r-k}) + rn - k. \end{aligned}$$

Now, since  $\hat{g}^{r-k} \in G_0(r, n - 1)$ , similarly to (36) we obtain

$$(38) \quad \text{fmaj}(\hat{g}) = \text{fmaj}(\hat{g}^{r-k}) + k(n - 1).$$

Since  $\text{fmaj}(\hat{g}) = \text{fmaj}(g)$ , the result follows from (37) and (38). The other two cases:  $i \leq r - k$  are similar and are thus left to the reader. ■

*Second proof of Theorem 11.2:* By Lemma A.2, and the decompositions (30) and (31) we obtain:

$$\sum_{g \in G(r, n)} t^{\text{fdes}(g)} q^{\text{fmaj}(g)} = \sum_{g \in G_0} t^{\text{fdes}(g)} q^{\text{fmaj}(g)} (1 + tq^n + t^2 q^{2n} + \dots + t^{r-1} q^{(r-1)n}),$$

and

$$\begin{aligned} \sum_{g \in G(r, p, n)} t^{\text{fdes}(g)} q^{\text{fmaj}(g)} &= \\ &= \sum_{g \in G_0} t^{\text{fdes}(g)} q^{\text{fmaj}(g)} (1 + tq^n + t^2 q^{2n} + \dots + t^{(d-1)} q^{(d-1)n}). \end{aligned}$$

It follows now that:

$$\sum_{g \in G(r,n)} t^{\text{fdes}(g)} q^{\text{fmaj}(g)} = \sum_{g \in G(r,p,n)} t^{\text{fdes}(g)} q^{\text{fmaj}(g)} (1 + t^d q^{dn} + t^{2d} q^{2dn} + \dots + t^{(p-1)d} q^{(p-1)dn}),$$

and Theorem A.1 ends the proof.  $\blacksquare$

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